

# Sheaf Theoretic Formulation of Entanglement

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## ABSTRACT

A formulation in terms of sheaf theoretic (or categorical) notions for quantum entanglement is given with direct experimental consequences. The notions from sheaf theory and category theory give structural theory, i.e., qualitative theory, as a candidate for quantum gravity. Its advantage is the following: it provides not only space-time background independent, but also scale independent. This theory is called the theory of temporal topos (or simply t-topos theory).

Keywords: Presheaf, contravariant functor, Grothendieck topology.

## INTRODUCTION

By presheafifying space, time and matter, a candidate theory for quantum gravity has been formulated (Kato 2003, 2004 and 2005) as the notion of t-topos. Let  $S$  be a site, i.e., a category with a Grothendieck topology, and let  $\hat{S}$  be the category of presheaves on  $S$ . (Gelfand and Manin 1996, Kashiwara and Schapira 2006, Kato 2006 for the needed mathematical backgrounds for t-topos.) Namely, for a presheaf  $m$ , object  $m(V)$  is in a category where an observation (or measurement) takes place between an observer  $P(V)$  and the observed  $m(V)$ . Then an observation is a morphism in the category (Kato 2004). On the other hand, we have the notions of decompositions of  $m$  into subsheaves  $m = \prod m_i$ , and of the object  $V$  in the t-site into covering decomposition as in Kato (2005). We have the projection morphisms

$$\prod m_i \longrightarrow m.$$

Notice: even when  $\mathbf{m}$  is defined for  $V$ , but  $m_i(V)$  may not be defined. That is, the observer  $P$  can measure  $\mathbf{m}$  over  $V$ , but possibly a decomposed micro subobject  $m_i$  may not be measured over the same  $V$  (Kato 2006). Conversely, for an observation morphism  $m_i(V) \longrightarrow P(V)$  for a subsheaf  $m_i$  by  $P$  over  $V$ , if  $m(V)$  exists, the composition of this morphism with the projection above gives a morphism from  $\mathbf{m}(V) = (\prod m_i)(V)$  to  $P(V)$ . Namely, global information does not provide local information on subsheaf  $m_i(V) \longrightarrow P(V)$ , but local information on a subsheaf partially provides global information through the projection morphism. If  $m(V)$  does not exist, no global information can be measured by  $P$  over  $V$  then associated presheaves  $\kappa$  and  $\tau$  of space and time to presheaf  $m$  have the following effects. For a morphism in the t-site

$$V \xrightarrow{g} U$$

we have, since  $\kappa$  and  $\tau$  are contravariant functors, the canonically induced morphism

$$\tau(V) \xleftarrow{\tau(g)} \tau(U),$$

and similarly for presheaf  $\kappa$  associated with space. Next, consider a covering

$$\{V_i \xrightarrow{f_i} V\}_{i=1,2}$$

of  $V$  in the sense of Kato (2005), Gelfand et al (1996) and Kashiwara et al (2006). Notice that we are considering a special covering with only two objects for the sake of simplicity. We let  $V' = V_1$  and  $V'' = V_2$ . Then we have the “restriction” morphism

$$\tau(V) \xrightarrow{\tau(f_1)} \tau(V'),$$

and similarly for  $V'' = V_2$ . The above “restriction” morphism should be understood as, for example, the time period  $\tau(V)$  for the observation of  $m(V)$  by the observer  $P(V)$  over the object (the generalized time period)  $V$  is restricted to a “shorter” time period  $\tau(V')$ . For a category with a Grothendieck topology, rather than a usual topological space, in general the set  $\text{Hom}_s(V, U)$  of morphisms from  $V$  to  $U$  consists of more than one element, where for a topological space such a set of morphisms consists of one element, i.e., the inclusion morphism. When  $\tau(V)$  precedes  $\tau(U)$ , e.g., measurement took place first during  $\tau(V)$ , then  $\tau(U)$ , there exists a morphism  $\varphi$  in  $\text{Hom}_s(V, U)$  indicating the induced morphism  $\tau(\varphi): \tau(U) \rightarrow \tau(V)$  is the linear order in the classical sense. Recall that presheaves associated with space and time are actually sheaves.

Namely, the following sequence for sheaf  $\tau$ , associated to time, is an exact sequence. For this material see Kato (2005), Gelfand et al (1996) and Kashiwara et al (2006).

$$\tau(V) \xrightarrow{\tau(f_j)} \prod_j \tau(V_j) \xrightleftharpoons[\tau(p_i)]{\tau(p_j)} \prod_{j,i} \tau(V_j \times V_i)$$

where, e.g.,  $p_j: V_j \times V_i \longrightarrow V_j$  is the projection.

Remark 1: In general, let  $\prod_{i \in I} m_i$  be a (micro-) decomposition a presheaf  $m$  in  $\hat{S}$  and let  $\{V_j \xrightarrow{f_j} V\}_{j \in J}$  be a (micro-) decomposition of an object  $V$  of the t-site, associated with a particle and a generalized time period, respectively. Those decompositions of  $m$  and  $V$  are said to be perfectly comparable when for all pairs of  $(i, j)$  in the index sets  $I$  and  $J$ , all the combinations  $m_i(V_j)$  of sub-presheaves  $m_i$  and covering family objects  $V_j$  are defined, i.e., all the  $m_i(V_j)$  are possibly observable states. However, further decompositions of sub-presheaves and further decompositions of generalized time periods in terms of further coverings, the comparability more likely breaks down. Consequently, some of pairs  $m_i(V_j)$  are not defined. Namely it does not define a state of  $m_i$  over the generalized time period  $V_j$ . Dark energy and dark matter may be able to be interpreted as the lack of comparability of the decompositions of presheaves and objects of the t-site in the micro decompositions (Kato, 2005). According to observations, measurement of mass in the universe indicates that further decompositions of presheaves and objects in t-site decrease the degree of compatibility of the measurable states  $m_i(V_j)$ , which is expected in the t-topos theoretic formulation. See Kato (2005) for ur-sub-Planck decompositions as inverse (projective) limits, which becomes relevant to the above consequences. See Kato G: “A Black Hole and t-Topos Entropy” (manuscript in preparation) for the t-topos interpretation of a black hole as a pseudo-category change. It may be premature to assert some consequences when those micro decompositions, “sufficiently close” to the inverse (projective) limits, happen to be compatible for all presheaves in the t-topos  $\hat{S}$  and for all objects in the t-site with respect to big bang.

Remark 2: For further micro decompositions of the matter represented by the presheaf  $m$ , then, assuming the existence of compatible pair  $m_i(V_j)$ , the associated generalized time period  $\tau(V_j)$  to  $m_i(V_j)$  may be short-lived (when a scaling is assigned). Recall that when  $\tau(V_j)$  is taken to the direct limit  $\varinjlim \tau(V_j)$  as in Eq. (7) in Kato (2005), the time period is shortest possible, where such an object is referred to as ur-sub-Planck decomposition of  $\tau(V)$ . As such a compatible pair approaches the direct limit, for a non-zero mass object, the

corresponding presheaf associated with the space  $\kappa(V_j)$  approaches the unbounded curvature (when a scaling is assigned). That is, the corresponding space and time associated to micro decomposed presheaves may be interpreted as “form-like” due to the short lived  $m_i(V_j)$  of non-zero mass particle  $m_i$ . Namely, shortness of  $\tau(V_j)$  and  $\kappa(V_j)$  associated with  $m_i$  creates the deformations of space and time. The details of this application in terms of t-topos will appear by Kato in “*A Black Hole and t-Topos Entropy*”.

## SHEAF THEORETIC ENTANGLEMENT

We will begin with the definition of the sheaf theoretic entanglement as defined in Kato (2004). We call such an entanglement “ur-entanglement” since the classical notion of the usual entanglement has been well established.

**Definition 2.1** Presheaves  $m$  and  $m'$  are said to be ur-entangled if they form a paired presheaf, i.e., they are defined on the same objects of the t-site  $S$ . We write such a paired presheaf as  $m^* = (m, m')$ .

One can also say that presheaves  $m$  and  $m'$  are ur-entangled when the paired presheaf  $m^* = (m, m')$  is defined on a t-subsite, where a subcategory is said to be a subsite if the subcategory satisfies the axioms for a site with respect to covering family. Namely, for an object  $V$  in the t-site  $S$ , we have the formula:

$$m^*(V) = (m, m')(V) = (m(V), m'(V)).$$

From this formula, when  $m$  is observed by  $P$  over  $V$ , then an ur-particle state  $m(V)$  is determined. Then the above formula implies that the state of  $m'$  is also determined, i.e., the ur-particle state  $m'(V)$ . It is crucial for applications that the ur-particle state of  $m'$  over  $V$  is determined even when  $m'$  is not observed over the generalized time period  $V$ . Compare with the applications made for double-slit interference in Kato and Tanaka (2006). One of the consequences is as follows: when an ur-entangled pair  $m^* = (m, m')$  is given, one of them is in (ur-) wave state if and only if the other is also in (ur-) wave state. For example,

**2.1 Prediction:** For an ur-entangled pair  $e^* = (e, e')$  of electrons as in Kato and Tanaka (2006), if  $e$  is passing through a single slit, then the other electron  $e'$  should show a particle state image on the screen in a typical double slit experiment setting even when  $e'$  is going through a double slit mask.

See also the report (Genovese, 2005) on recent hidden variable approaches. From the definition of ur-entangled pair  $m^* = (m, m')$ , for example,  $m$  and  $m'$  can belong to different light-cones. (See Kato (2005) for the definition of a light cone in the t-topos sense.)

**Final Remark:** The fundamental approach in t-topos theory is the (pre)sheafification of space, time and matter as in Kato (2003). Then the methods from category and sheaf theory enable t-topos to explain the particle-wave

duality, uncertainty principle, entanglement almost canonically as in (Kato, 2004, 2005, 2008 and Kato et al. submitted). Our forthcoming project is to study the t-topos theoretic aspects and consequences for black holes (Kato, "Black Hole and t-Topos Entropy").

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